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Framework for Traffic Congestion Prediction

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Abstract— Traffic Congestion is a complex dilemma facing most major cities. It has undergone a lot of research since the early 80s in an attempt to predict traffic in the short-term. Recently, Intelligent Transportation Systems (ITS) became an integral part of traffic research which helped in modeling and forecasting traffic conditions. In this paper, two frameworks for traffic congestion prediction are proposed. The first framework is based on NeuroFuzzy model which is well surveyed in traffic literature. The second framework is based on Hidden Markov Models (HMM) which is rarely used in traffic prediction. The methods are used to define traffic congestion during morning rush hours. The results of the two methods are compared.

The empirical evaluation is based on a UK dataset which is provided by the UK Department of Transport. The data is a year on year statistics from 2009 to date and is available in a monthly ".CSV" files. It was collected using loop detectors and consolidated every 15 minutes for various links of the UK motorways.

Index Terms— NeuroFuzzy, Hidden Markov Models, Traffic Congestion Prediction, Empirical Evaluation

1 Introduction

Traffic congestion has become an integral part of today's modern life. It forces people to plan additional time whether commuting to work, or traveling for other purposes. It results in longer trip times, lower air quality, and increased fuel wastage which in turn affect the overall quality of life. Therefore, governments, universities, and advanced research are attempting to tackle this problem or at least ease its adverse effects using intelligent transportation systems (ITS). A major part of the ITS is traffic forecasting based on realtime data to enable traffic decision makers to make the right decisions.

There are various research methods used in the field of traffic prediction such as deterministic methods, non-deterministic approaches, and stochastic techniques. In this work, two frameworks are proposed for traffic congestion prediction during the morning rush hour. The first framework is based on NeuroFuzzy technique which is well surveyed in traffic literature. The second framework is based on HMM which is rarely used in traffic prediction due to its complex nature. The results of the two methods are compared.

The organization of this paper starts with a review perspective of the recent research followed by the introduction of the realtime dataset to be used in the empirical evaluation. Next, the theory behind this research is discussed. Sequentially, the results and discussions, and the concluding remarks are presented.

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2 Review of Existing Techniques

Short-term traffic forecasting is a challenging research opportunity. It attracts various researchers using a multitude of methods to attempt forecasting different traffic parameters. In a review paper, Vlahogianni et. al. [1] reviewed 10 challenging research opportunities in the field of ITS focusing on forecasting problem in ITS. Recently, Hashemi and Abdelghany [2] developed a real-time traffic state prediction based on closed loop rolling horizon. In their approach, some real-time system deficiencies such as limited prediction accuracy, decision making latency, and partial coverage of the managed area. In another paper, ELHenawy and Rakha [3] detected congestion using two-component mixture model. One is based on free-flow speed distribution and the other is based on congestion speed distribution. The model was calibrated and a threshold was identified where congestion is detected if below the threshold. Dong et. al. [4] proposed a spatio-temporal approach for freeway traffic flow prediction. Their approach shows 5% results improvement over the standard autoregressive integrated moving average (ARIMA) model. Yuan et. al. [5] suggested a new model for traffic state estimation based on Lagrangian-space and Kalman Filtering (KF). Their approach provided more accurate numerical results compared to traditional methods in the same coordinate system. Tao et. al. [6] developed a time-space threshold vector error correction (TS-TVEC) model for short-term traffic state prediction. The statistical model overcomes unknown structural changes in time series. Jeong et. al. [7] presented an online learning weighted support-vector regression (OLWSVR) for short-term traffic flow predictions. The model performance was superior to well-known prediction methods. Several other research papers focused on prediction are available; (see [8] [9] [10] [11] and [12]).

Having reviewed various prediction methods, the focus now is on stochastic and statistical methods. Recently, Guo and colleagues [13][14] proposed stochastic autoregressive algorithms for predicting short-term traffic condition under uncertainty. In another study by Turochy [15], he coupled the nearest neighbor of nonparametric regression with condition monitoring. This detects the deviation of current traffic condition from the expected condition based on historical data. Tchrakian et. al. [16] proposed two approaches, one based on spectral analysis and the other based on weighted average to predict short-term traffic flow. In a different study, Xie and Zhao [17] proposed Gaussian Processes (GPs) model for short-term traffic flow forecasting. He showed advantage over Support Vector Machine (SVM) and (ARIMA) models. Sun and Xu [18] introduced variational infinite Gaussian mixture model to the problem of traffic flow prediction. The approach as compared to other approaches showed better effectiveness. In a different article, Fei et. al. [19] introduced Bayesian inference-based dynamic linear model (DLM) integrated into adaptive control framework to predict online short-term travel time. Empirical evaluation proved that the method is accurate and reliable. There is a number of research articles on short -term traffic prediction using statistical methods (see [20] [21] [22] [23] and [24]).

Zhang et. al. [25] introduced a Fuzzy Wavelet NN algorithm. The algorithm is optimized by Quantum Particle Swarm Optimization (QPSO) algorithm. In their paper, Li et. al. [26] used Feed-Forward Neural Network (FFNN) for traffic Prediction. Li [27] used dynamic fuzzy neural network (D-FNN) for traffic flow prediction. Kazemi and Abdollahzade [28] developed local linear neuro-fuzzy model that is trained offline and adapted to online data using weighted least squares. In another article, Celikoglu [29] introduced an NN for real-time mapping of tra□c density in conjunction with a macroscopic tra□c flow model. Further reading available in papers (see [30] [31] [32] and [33]).

3 DATASET

This research is based on data collected from the Highways in England. The network is composed of 4400 miles of major motorways in England and accounts for only 2% of all England's roads [34]. England's Highway Agency made traffic data available for the public in monthly comma separated files from 2009 to date. Each monthly file contains roughly 7 million records of traffic flow data. As shown in the sample Table 1, the data is averaged every 15 minutes for all the junctions resulting in 96 readings per junction per day (2976 readings per junction per month). Table 2 shows the explanation of the fields in Table 1.

Table 1: Sample Traffic Data

LinkRef	Date	TimePeriod	AverageJT	AverageSpeed	DataQuality	LinkLength	Flow
AL1260	01/12/2014	0	114	82.11	4	2.6	10.75
AL1260	01/12/2014	1	113.71	82.31	2	2.6	10.25
AL1260	01/12/2014	2	127.19	73.59	2	2.6	8.5
AL1260	01/12/2014	3	110.75	84.51	2	2.6	6.5
AL1260	01/12/2014	4	124.7	75.06	2	2.6	5.5
AL1260	01/12/2014	5	124.31	75.3	2	2.6	5
AL1260	01/12/2014	6	123.76	75.63	4	2.6	5.5
AL1260	01/12/2014	7	121.26	77.19	4	2.6	5.5
AL1260	01/12/2014	8	115.38	81.12	4	2.6	5.75
AL1260	01/12/2014	9	120.23	77.85	4	2.6	6.5
AL1260	01/12/2014	10	132.81	70.48	2	2.6	7.5
AL1260	01/12/2014	11	115.03	81.37	2	2.6	6.5
AL1260	01/12/2014	12	122.49	76.41	2	2.6	7
AL1260	01/12/2014	13	114.28	81.9	2	2.6	11
AL1260	01/12/2014	14	120.45	77.71	2	2.6	14.5

Table 2: Meaning of Column Headers

Variable name	Variable description
LinkRef	A unique alphanumeric link id representing a junction to junction link.
Date	Date of travel.
TimePeriod	One of 96 15-minute intervals in the day (0-95 where 0 indicates 00:00 to 00:15).
AverageJT	The average journey time to travel across the LinkRef in seconds.
AverageSpeed	The average speed (km/h) of vehicles entering the link within a given 15-minute time period.
DataQuality	Indicator showing the quality of the journey time data for the link and time period. 1 indicates the highest quality data and 5 the lowest. See below for detailed description: 1 = Observed or vertically1 in-filled data with a good spatial match2 to the link 2 = Observed or vertically in-filled data with a poor spatial match to the link. 3 = Horizontally3 in-filled data with a good spatial match to the link. 4 = Horizontally in-filled data with a poor spatial match to the link. 5 = No observed data so data are in-filled using free-flow data.
LinkLength	The length of the link (km).
Flow	An average of the observed flow for the link, time period and day type.

Since the interest here is predicting traffic condition using non-deterministic models, the quality of the data is utmost importance. Hence, data mining techniques were used to extract a suitable junction data for the purpose of this research.

For example, a profile of a certain junction does not have any sizable congestion pattern or another junction profile that only contain slow speeds which could bias the study towards urban traffic instead of highway. Therefore, smart routines were developed to qualify the junctions based on the data profile available to suit the study at hand.

A junction called "AL2701" representing the A45 between A46 and A46 is chosen for this research. A standardization of the units is applied to allow for the calculation of additional variables and solid analysis. According to Kerner's [35] three phase traffic theory, the fundamental flow-density relationship and the fundamental speed-flow-density relationships are shown in Fig. 1 (a) and (b) respectively. They represent the full profile of traffic speed, flow, and density relationships. That is, any traffic pattern may have a part of the profile in the fundamental relationships graphs.

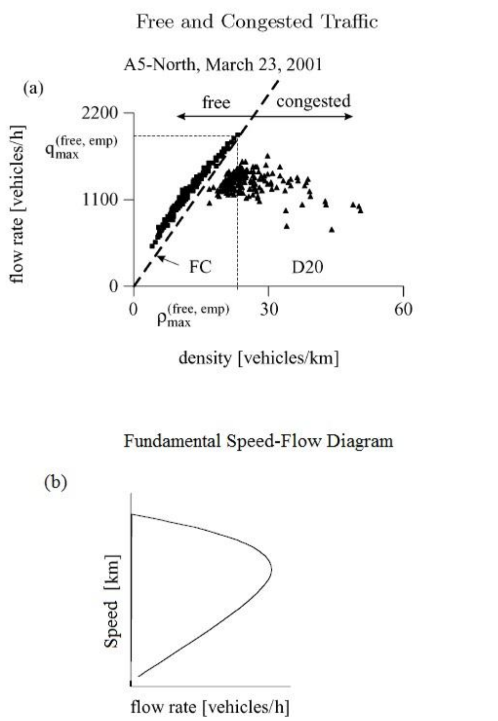

Fig. 1
Fundamental Flow-Density and Speed-Flow Relationships

4 METHODS

The aim of this section is to develop two prediction models for traffic congestion from historical empirical data. The first model is NeuroFuzzy based prediction model. The second model is HMM based prediction model. The results of the two methods are compared to each other.

4.1 NeurFuzzy Model

Fuzzy systems map crisp inputs nonlinearly into crisp output through four main stages: fuzzifer, rules, inference engine, and defuzzifier [36]. The inputs are converted into fuzzy sets using membership functions through the fuzzifier stage. The next step is the inference; it is made based on a set of rules. Finally, the output is generated using output membership functions through the defuzzification stage. The membership functions are used to map the non-fuzzy data into fuzzy sets and vise versa. The rules of a fuzzy system are simply IF-THEN statements. For example, IF speed is less than 15 and density is more than 40 THEN the traffic is congested.

Artificial Neural Networks (ANN) is analogical to human brain. They are composed of neurons connected to each other by links. Those links carry certain weights. If the network does not provide the required output then the weights of the links are adjusted accordingly through the learning process. Since fuzzy can inference results from imprecise or uncertain data, and neural networks can recognize patterns by updating its weights. A hybrid algorithm of neuro-fuzzy is suitable for predicting traffic congestion from historical data.

The model implemented in this study is a Sugeno based Adaptive Neuro Fuzzy Inference System (ANFIS). The model is implemented in MATLAB according to the algorithm shown in Fig. 3 and Fig. 4 (a).

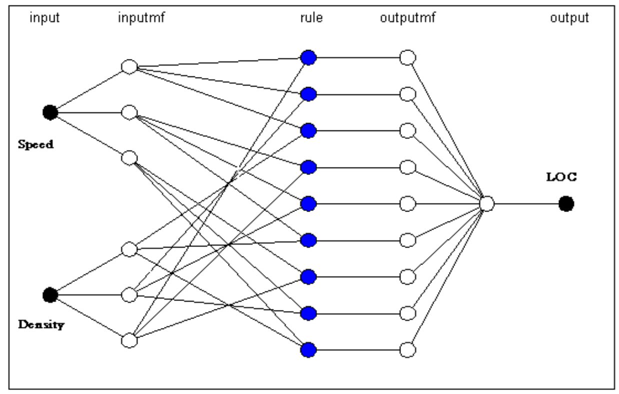

Fig. 3 Neuro-Fuzzy Model

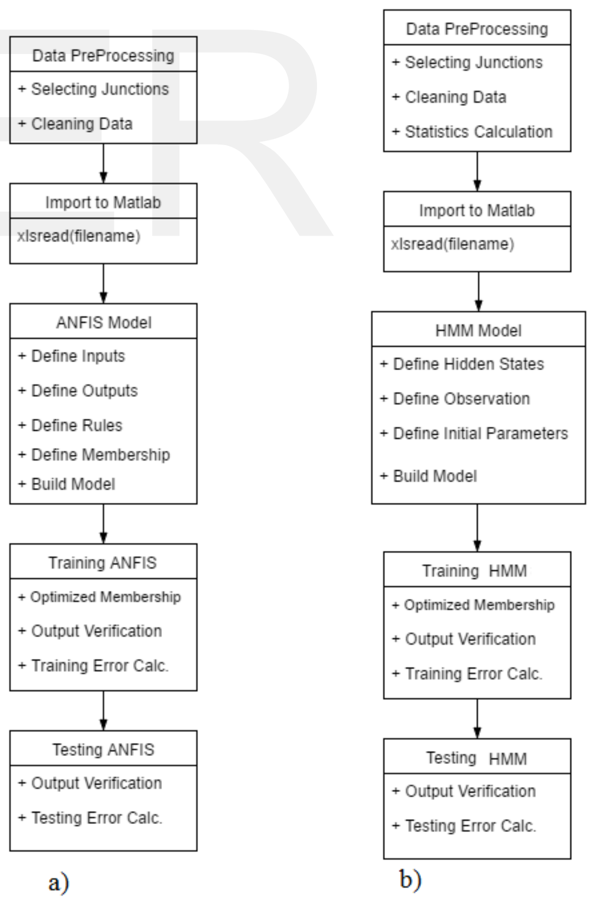

Fig. 4: High Level UML for ANFIS and HMM

The data under consideration is first extracted from the dataset. It is then cleaned from outliers where a data point is considered an outlier if it is lower than one sixth of the sum of the two points around it. If so, it is replaced by the average of those two points. Since the top speed on UK motorways is 70 miles/hr which is roughly 110 Km/hr, any data point that is greater than 120 Km/hr is considered an outlier. After data cleaning, the processed data is imported into MATLAB where the ANFIS model is built. The inputs, output, membership functions, and fuzzy rules are defined according to the model below:

- Two Inputs: speed and density, each of three levels indicating
  - Speed levels: Slow, Medium, Fast
  - Density levels: Low, Medium, High

- One Output: Level of Congestion (LOC)

- Nine Rules:
  - Two rules representing free flow traffic.
  - Two rules representing slow moving traffic.
  - One rule representing mild congestion.
  - Two rules representing heavy congestion.
  - Two rules representing serious congestion.

- Trapezoidal membership functions for the fuzzy system.

- Number of training epochs = 100

The model is trained using 70% of the data while the remaining 30% is reserved for testing.

4.2 Hidden Markov Model

Hidden Markov Model (HMM) is a probabilistic method used in many state recognition and classification applications [37]. However, it is rarely used in traffic prediction although traffic is a stochastic process. In HMM, there are the observation and the hidden states of the system where the observation provides information regarding the state of the system. Since the speed can be measured while the traffic condition is unknown, the speed is considered as the observation and the traffic condition is considered the state according to HMM structure. In other words, Rabiner [38] stated that a hidden underlying stochastic process which can be observed through another observed stochastic process conforms to HMM structure. The dynamics of such processes are captured into State and Emission matrices.

Three important problems face any HMM structure, which are discussed by Rabiner [38]. Those are:

- What are the initial model parameters that maximize the probability of a certain observation, given an observation sequence?
- What is the probability of a certain observation, given initial model parameters and an observation sequence?
- What is the optimal state transition sequence? Given a set of observation sequence and the HMM parameters.

Dempester et. al. [39] used Expectation Maximization (EM) to find the maximum likelihood of the model initial parameters. Baum-Welsh iterative training algorithm [40] is used to optimize the model parameters. Moreover, Viterbi algorithm is used to find the optimal state sequence associated with a given observation sequence. That is predict the optimal state sequence using path backtracking.

The implementation of HMM requires an additional step in data preparation more than ANFIS model. That is, preparing statistics such as the mean and standard deviation since statistics provide a measure of trend. Mean measures the central tendency of the data while standard deviation measures the spread of the data around the central tendency. The data is imported into MATLAB and Kevin Murphy HMM toolbox was used to generate the model as shown in Fig. 4 (b). Again 70% of the data were used for training and 30% were saved for testing.

To use HMM for traffic prediction, the traffic data must be clustered. Clustering is completed using K-means and the number of clusters is evaluated using sum of squares. Once clustering is completed, a number of HMMs equal to the number of clusters is trained. Using the log-likelihood, a test vector belongs to a certain cluster depending on the value of the log-likelihood resulting from passing the vector on all the HMMs available.

5 RESULTS AND DISCUSSIONS

The neuro-fuzzy model implemented above predicts the state of congestion ranging from free flow traffic to serious congestion. The algorithm is trained using 70% of the data of a certain junction. It is then tested against the training data as well as tested against the remaining 30% of the data. It has also been tested with data from different junctions. The results of such experiments are shown in Fig. 5. It represents the level of congestion LOC (Human decision in blue) and the LOC (network decision in red). The prediction error produced by the ANFIS model is 11%.

To use HMM, K-means clustering was used to find the traffic clusters as shown in Fig. 5. The line with a different color in each cluster represents the cluster average speed trend. Again 70% of the available data was used for training to find the optimal HMM parameters. That is, finding the optimal state transition and emission matrices. Once the optimal parameters are available, HMM can be used to classify and predict the traffic condition of the 30% testing data. The traffic patterns are classified using Log-Likelihood showing a classification error less than 15%. While the traffic states are predicted using Viterbi Algorithm showing a prediction error less than 10%.

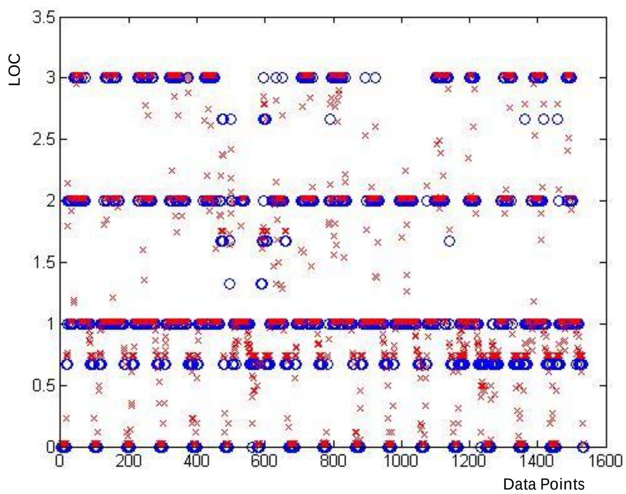

Fig. 5: ANFIS model prediction result

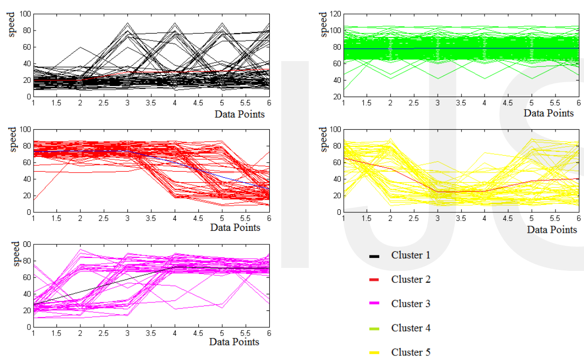

Fig. 5: K-means Clusters

6 Conclusion

This paper is discussing two different approaches for traffic congestion prediction. It uses NeuroFuzzy approach showing 11% prediction error. It also uses HMM to predict traffic showing a prediction error of 10%. This confirms the suitability of HMM for traffic congestion prediction. This is due to the fact that HMM is a stochastic method and traffic is stochastic in nature. HMM utilizes statistics such as mean speed and standard deviation to predict traffic conditions.

The dynamics of traffic are captured into the HMM state transition probability matrix and the observations of speed are captured into the emission probability matrix. The trend of traffic and variations are captured through the statistics. For a sequence of traffic speed observations, HMM estimates the most likely sequence of traffic states using Viterbi Algorithm path backtracking.

The purpose of this research is to assist traffic departments in their short-term decision making. Whether to send traffic officers to a particular location, use variable speed signs upstream to divert traffic into alternative roads, or increase the number of recovery vehicles in a particular stretch of road at certain times. It also supports traffic management and improvement strategies on the longer term such as as ramp metering. Such forward looking approaches can only be applied if prediction is available.

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